RAMAKRISHNA MISSION VIDYAMANDIRA
(Residential Autonomous College affiliated to University of Calcutta)
B.A./B.Sc. FIFTH SEMESTER TAKE-HOME TEST/ASSIGNMENT, MARCH 2021
THIRD YEAR [BATCH 2018-21]
Date : 13/03/2021 MATHEMATICS (Honours)
Time : 11am – 3pm Paper : V Full Marks : 100

Instructions to the Candidates

- Write your **College Roll No, Year, Subject & Paper Number** on the top of the **Answer Script**.
- Write your **Name, College Roll No, Year, Subject & Paper Number** on the **text box of your e-mail**.
- Read the instructions given at the beginning of each group/unit carefully.
- Only handwritten (by blue/black pen) answer-scripts will be permitted.
- Try to answer all the questions of a single group/unit at the same place.
- All the pages of your answer scripts must be numbered serially by hand.
- In the last page of your answer-scripts, please mention the total number of pages written so that we can verify it with that of the scanned copy of the scripts sent by you.
- For an easy scanning of the answer scripts and also for getting better image, students are advised to write the answers in single side and they must give a minimum 1 inch margin at the left side of each paper.
- After the completion of the exam, scan the entire answer script by using Clear Scan: Indy Mobile App OR any other Scanner device and make a **single PDF file (Named as your College Roll No)** and send it to [redacted]

Group - A (Abstract Algebra)

Answer all the questions from this group. Maximum you can obtain in this group is 50.

1. An element x of a group is called a square if $x = y^2$ for some y in the group. Suppose H is a subgroup of an abelian group G. If every element of H is a square and every element of G/H is a square then show that every element of G is a square. [3]
2. Characterize all simple commutative groups. [2]
3. Suppose G is a finite abelian group and G has no element of order 2. Show that the mapping $f : G \to G$ defined by $f(x) = x^2, \forall x \in G$ is an isomorphism. What happens if G is an infinite abelian group? [4+2]
4. Find all automorphisms of the group $\mathbb{Z}_{12}$. [4]
5. Can a group of order 8 contain 7 elements of order 2? Justify. [3]
6. Give an example of an infinite abelian group with exactly 6 elements of finite order. Justify your answer. [3]
7. Consider the multiplicative groups $(\mathbb{R}^*, \cdot)$ and $(\mathbb{R}^+, \cdot)$, where $\mathbb{R}^* = \{x \in \mathbb{R} : x \neq 0\}$ and $\mathbb{R}^+ = \{x \in \mathbb{R} : x > 0\}$. Show that $\mathbb{R}^*$ is an internal direct product of $\mathbb{R}^+$ and the subgroup $\{1, -1\}$. [3]
8. Can you express the group $(\mathbb{Q}, +)$ as an internal direct product of two proper subgroups? Justify your answer. [3]
9. Prove that no group of order 12 is simple. [5]
10. Prove that a group of order 85 is cyclic. [5]

11. Does there exist a ring epimorphism from $\mathbb{R}$ to $\mathbb{Z}$? Justify. [3]

12. Find all the units of $\mathbb{Z}[i]$. Find all the associates of $3-2i$ in $\mathbb{Z}[i]$. [4]

13. Prove that $2+i\sqrt{5}$ is irreducible but not prime in $\mathbb{Z}[i\sqrt{5}]$. [3 + 3]

14. Show that the field $\mathbb{Q}$ has no proper subfield. [3]

15. Let $I = \{(m,n) \in \mathbb{Z}\times\mathbb{Z} : 3|n\}$. Show that I is a maximal ideal of $\mathbb{Z}\times\mathbb{Z}$. [4]

16. Show that $4\mathbb{Z}$ is a maximal ideal of $2\mathbb{Z}$ but not a prime ideal of $2\mathbb{Z}$. [3]

Group - B (Multivariable Calculus)

Answer any 3 questions from question nos. 17-21 in this group. [3 x 10 = 30 marks]

17. (a) Check whether the simultaneous and repeated limits exist for the function f given below, as x and y both tend to 0. [4]

$$f(x,y) = \begin{cases} y\sin\left(\frac{1}{x}\right) + \frac{xy}{x^2+y^2}, & \text{if } x \neq 0 \\ 0, & \text{otherwise.} \end{cases}$$

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ and $g : \mathbb{R}^3 \to \mathbb{R}^2$ be given by

$$f(x,y) = \left(e^{x+2y}, \sin(2x+y)\right) \quad \text{and} \quad g(u,v,w) = (u+2v^2+3w^3, 2v-u^2)$$

Further, if $h(u,v,w) = f\circ g(u,v,w)$, compute the Jacobian matrices $Df(x,y), Dg(u,v,w)$ and $Dh(1,-1,1)$. [6]

18. (a) Let $S \subseteq \mathbb{R}^2$ be compact and $f : S \to \mathbb{R}^2$ be continuous and one-one on S. Show that the inverse $f^{-1} : f(S) \to S$ is continuous. [2]

(b) f is continuous iff for each subset $E \subseteq \mathbb{R}^3$ we have $\overline{f^{-1}(E)} \subseteq f^{-1}(\overline{E})$. (i.e. closure of the preimage is contained in preimage of the closure). [3]

(c) Let S be a nonempty subset of $\mathbb{R}^n$. The distance of a point $x \in \mathbb{R}^n$ from a set S is defined by $d(x,S) = \inf\{||x-y|| : y \in S\}$. Show that if S is compact, then $\exists y_0 \in S$ such that $d(x,S) = d(x,y_0)$. [5]

19. Let $f : \mathbb{R}^2 \setminus \{(0,0)\} \to \mathbb{R}$ be smooth and homogeneous of degree d. Prove that [3+4+3]

(a) if $d = 0$, then f is bounded. Also, prove that f extends to be continuous at $(0,0)$ iff f is a constant function.

(b) if $d > 0$, then f is continuous everywhere if we define $f(0,0) = 0$. Also, prove that if $d < 0$ then we can not make f continuous at $(0,0)$.

(c) If f is homogeneous of degree 1 and satisfies $f(-x) = -f(x)$ and $f(0) = 0$, prove that $\frac{\partial f}{\partial x_j}$ is not continuous at $(0,0)$ unless it is a constant function.

20. (a) If $\frac{x^2}{a+u} + \frac{y^2}{b+u} + \frac{z^2}{c+u} = 1$, prove that [6]

$$\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) u = 2\left(x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\right) u$$

(b) Find the Taylor expansion about $(1, \pi/2)$ of $\sin x$ and $\sin y$ upto second degree terms. [4]

21. (a) Find the shortest distance from the origin to the surface $z^2 - xy = 1$. [4]

(b) Let $f : \mathbb{R}^2 \to \mathbb{R}$ be such that all first, second and third order partial derivatives of f exists and is bounded over the entire domain. Will the mixed derivatives $f_{xy}(a,b)$ and $f_{yx}(a,b)$ be equal for any arbitrary point $(a,b) \in \mathbb{R}^2$? Justify. (No marks will be awarded for wrong or no justification.) [6]

Group - C (Riemann Integration and Bounded Variation)

Answer any 2 questions from question nos. 22-24 in this group. [2 x 10 = 20 marks]

22. (a) Show that the function $f(x) = [x], 1 \leq x \leq 3$ is function of bounded variation on $[1,3]$. Find its variation function on $[1,3]$. Express f as a difference of two monotone non-decreasing functions. [6]

(b) Test whether the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} \sqrt{x}\sin\left(\frac{1}{x}\right) & , 0 < x \leq 1 \\ 0 & , x = 0 \end{cases} .$$

is a function of bounded variation on $[0,1]$. [4]

23. (a) Test the Riemann integrability of the function $f : [0,1] \to \mathbb{R}$ defined by [5]

$$f(x) = \begin{cases} \sqrt{1-x^2} & , x \in [0,1] \cap \mathbb{Q} \\ 1 - x & , x \in [0,1] \cap \mathbb{Q}^c \end{cases}$$

(b) Show that [5]

$$\frac{\sqrt{3}}{8} \leq \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sin x}{x} dx \leq \frac{\sqrt{2}}{6}$$

24. (a) Let $f : [1,3] \to \mathbb{R}$ be defined by [6]

$$f(x) = \begin{cases} 1 & , 1 \leq x < 2 \\ 2 & , 2 \leq x \leq 3. \end{cases}$$

State with reasons :

i. Whether f is Riemann integrable on [1,3].

ii. Whether the formula $\int_a^b f(x)dx = (b-a)f(\xi)$ for some $\xi \in [a,b]$ is true.

iii. Whether the fundamental theorem of integral calculus is applicable to f on [1,3].

(b) Calculate [4]

i. $\lim\limits_{x \to 4} \frac{1}{x-4} \int_4^x e^{\sqrt{1+t^2}} dt.$

ii. $\lim\limits_{x \to 0} \frac{x}{1-e^{x^2}} \int_0^x e^{t^2} dt.$

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